Recent Advances in Applications of Instructional Technology for Linear Algebra and Numerical Algorithms

Dr. Jeffery J. Boats

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1. Abstract

Some of the most dynamic advances in the use of technology in linear algebra and numerical analysis are taking place, not in the research laboratory, but in the classroom. In fact, the very notion of the "conventional classroom" is being challenged, with computer screens playing the role of blackboards, web sites augmenting or even replacing textbooks, and instructors teaching from hundreds if not thousands of miles away.

This lecture expounded on some of the more recent and impressive developments in instructional technology, as it pertains to linear algebra and numerical algorithms. This included a synopsis of:

- ◇ Tutorial web sites, such as the ILAW site at the University of Calgary;
- ◇ Educational software used in laboratories;
- ◇ Textbooks with innovative approaches to exercises, implementing computer tutorials and new applications of linear algebra and numerical analysis;
- ◇ Recent research and development in distance learning.

The audience was provided with demonstrations and many informative references.

2. A Historical Look at the Future

It's been fashionable for some time in science education circles to discuss the advent of new technologies, and to debate whether these new tools are more a catalyst to the learning process or a hinderance. Today, when we speak of technology in the mathematics classroom, we generally refer to digital technology, ranging from the graphing calculator to networks of interconnected computers with the latest software packages for both communication and computation. But the following reference demonstrates our modern Socratic diaologue to be simply the latest incarnation of a historic argument.

"There are two very different postulates ... that attempt to evaluate radio and television as educational resources. One postulate asserts that these resources are valuable only if they can do what is impossible for the classroom teacher ... or if they can do what the teacher can do ... better. The other postulate asserts that the new resources are [intrinsically] valuable because they supply ... a different kind of teaching ..." **[WC]**

When this article first appeared in 1954, computers still ran on gigantic tape spools and filled entire rooms, and were not practical for computational use in the classroom. But radio and television offered an enhanced communication between professional educators, and could serve as an augmentation to traditional pedagogies.

How far we have come. Over forty years of technological advances and educational research have given birth to generation after generation of ever-evolving pedagogical tools. The presence of technology in the classroom is no longer debated – it is now *advertised*. It is a primary marketing tool for many upstart universities, as well as the key ingredient in the revitalization of many well-established universities. Let the last lingering doubts over the merit of computers in the mathematics classroom be relieved.

"Asking whether computers are useful in teaching mathematics is like asking whether blackboards are useful in teaching mathematics. The answer is: it depends on how you use them." **[RS]**

There is now a different set of essential questions to be answered.

◇ What technology is available?
◇ When should it be implemented?
◇ How should it be implemented?

Our interest is not merely how these questions are answered in general, but how they are answered in the context of modern pedagogies used in teaching computer-intensive courses in undergraduate mathematics, namely linear algebra, numerical analysis, and related topics.

3. Essential Terminology

This discussion deals primarily with *computer-assisted instruction* (CAI), where technology is used to facilitate or augment the instruction of a human teacher. This contrasts *computer-managed instruction* (CMI), which refers to automation – technology used as a medium for automated testing, distribution of course materials, etc. Computer-managed instruction is generally not an intrinsic part of the instruction process.

On the other hand, computer-assisted instruction is, by definition, part of the teaching scheme, and it takes two forms. *Active technology* will refer to the use of technology in such a way that it is essential during the primary instruction. This can mean that the instruction takes place through the technology, using it as a medium, or it could mean that the instructor and/or the students use it actively in the classroom as the class is in progress. Meanwhile, *passive technology* is not used during the primary instruction, but rather as a supplemental tool or resource outside of class time. One form of active technology which is growing ever more popular as technology improves is *distance learning*. Here, computers are used as the medium for instruction, as the student(s) and instructor(s) are geographically too far apart for a normal classroom setting to be convened.

The technology itself it called *instructional technology*, and can be utilized in many distinct ways. *Managerial* instructional technology is that which is used in CMI. *Computational* technology refers to software used in computation, but also includes the software and hardware utilized in program-writing assignments. *Communicative* technology is that used to contact the instructor, other students, or outside (human) sources, while *reference* technology makes contact with electronically-posted information, such as that in databases, webpages, etc.

One main focus of this lecture was another form of instructional technology, *tutorial* technology. A computer tutorial is a guided exercise format designed to stand alone as a secondary source of instruction. A computer tutorial is distinguished from other computer exercises in that:

◇ it consists of a series of leading steps which build gradually to the discovery of a specific concept;

◇ the questions include directed opportunities for the students to verbally analyze the progress and purpose of the steps;

◇ the conclusion of the exercise features an opportunity for the students to apply what they have learned.

4. The Passive Option

Passive technology is easier to create and utilize, as it does not involve direct interaction between instructor and student. While databases of programs and software are commonplace, the most recent addition to the arsenal of passive technology is the World Wide Web, which makes properly posted information instantly accessable worldwide.

Most university teachers can easily create web pages providing easy access to course materials for students in their courses. An excellent example of this is the linear algebra page [**SW**] of Gilbert Strang, a professor of mathematics at the Massachusettes Institute of Technology and author of a linear algebra textbook [**GS**]. The key in setting up reference pages for courses is using application software which make the downloading of information for convenient and in more usable forms, though making certain the utilized technology is available to all students.

Many references are not specific to individual courses or universities. Some offer assistance to the general public or the private consumer. An excellent example of this sort of reference technology is the SOSMath website [**SO**], created and maintained by Nancy Marcus, Amin Khamsi, and Helmut Knaust of the University of Texas at El Paso. It offers information and lessons in the basic concepts of many high school and undergraduate mathematics courses, and is easily accessed from the internet; it can also be purchased for installation on a personal computer. It uses html, but does not yet make use of JavaScript.

The ATLAST website [**AW**] offers MATLAB programs which inhabit an ingenious workbook [**AB**] of linear algebra projects tutorial exercises. The workbook was assembled by a committee of over one hundred dedicated professors at nearly as many universities, and is an excellent supplement to a standard linear algebra textbook.

Websites like AllExperts [**AE**] and Ask Dr. Math [**DM**] are communication technologies, providing students with an outlet for asking specific questions to human sources. Be warned, however, that there does not seem to be a rigorous screening regimen for the site "experts."

The latest and perhaps most impressive passive technology is the interactive textbook. The Interactive Linear Algebra on the Web project [**IL**], or ILAW for short, is an on-line textbook written by Claude LaFlamme and W. Keith Nicholson of the University of Calgary. It features standard lessons augmented by interactive demonstrations and tutorials, as well as many projects focusing on current applications in science and engineering, many of which make creative use of JavaScript to provide students with a venue for guided experimentation. It is still a work in progress, but instructors may write the authors for permission to sample the up-and-running portions of the text.

5. The Active Ideal

The recent hotbed of education research, as it pertains to active technology, is distance learning. The growing availability and increasing capability of the internet is opening up opportunities for universities to engage in outreach programs, teaching to students at remote locations. As this is a new form of instruction, it has become an important issue to determine how such programs can and ought be run.

It goes without saying that this form of instruction demands rigorous preparation, with a level of organization and administration far beyond that of the traditional classroom. At a distance, such normally trivial concerns as contact between students and faculty become highly non-trivial. The primary concerns of the distance instructor may be summarized in the following problems to be addressed.

◇ How is a lecture delivered effectively over a distance, perhaps to multiple locations simultaneously?

◇ How can students, during the live session, respond instantaneously to the instructor, establishing an active learning environment?

◇ How can students from different locations interact with each other, as students do in traditional classrooms? (Can cooperative learning be used in this setting?)

The technology to address all of these problems exists, though much of it is not widely available and affordable yet. Ideally, a real-time audio-visual display of the lecturing instructor could appear on the student's screen in a frame. Simultaneously, in an adjacent frame, graphics displays featuring graphs, diagrams, and computer algorithms and outputs could appear in time with the lecture. Real-time chat could allow student-to-student discussion during the class, with a special script denoting questions asked of the instructor. Automated assessment and testing could be easily implemented, and course materials, assignments, and source codes could be delivered by e-mail or fax. The existing technology allows countless variations of creative pedagogy. But it's a very lucky professor whose university can meet his every technological wish.

6. The Active Ideal Meets Reality

At the Instituto Tecnologico de Monterrey in Mexico, an introductory linear algebra course was offered as a distance education course, as part of a teacher training program for teachers working at several campuses of a technological university spread throughout Mexico. The teachers were enrolled in a “Masters in Education” program with specialization in mathematics. Most of them were comfortable with the numerical or algorithmic aspects of linear algebra concerning matrices and linear equations, but had little or no experience with the more abstract concepts, such as vector spaces and linear transformations.

With the help of an instructional designer, this distance education course was structured as follows:

◇ Students were put into virtual groups of 3 or 4 members. It was ensured that there would be at least two different campuses represented in each group. This meant that everybody had to communicate with at least one person in the group in a non-visual manner.

◇ No lectures were given. There were 8 bi-weekly satellite sessions that were held after the groups had handed in their homework; the interactive nature of the course was encapsulated in these sessions. Each campus had a TV screen through which the students viewed the session. Written communication between the campuses and the instructor was achieved through SIR (Sistema Interactiva Remota) during these sessions. In this manner, the students were able to ask and answer questions or make comments. Satellite sessions served as a forum to discuss different solutions to a given problem, to clarify doubts, to talk about common mistakes and discuss interesting aspects of the topics in question.

◇ Regular homework and projects were assigned, and the participating students were expected to read the corresponding sections from the textbook, discuss the homework problems with each other, and turn in a common group report. While they were working on their homework and projects, they were expected to communicate with each other via e-mail or discussion groups on the internet, sending a copy of each message to the instructor. This way, the instructor (Asuman Oktaç) was able to follow the discussion, interfering only when necessary, for the purpose of guiding the participants in a relevant direction.

◇ When necessary, internet chat sessions were arranged to provide real-time communication. These served the purpose of asking questions to the instructor or clarifying points of doubt among the group members concerning certain aspects of their projects.

It almost goes without saying that the logistics of such an operation demand meticulous organization, not only in terms of pre-class preparation, but also while class is in progress. During the real-time satellite sessions, a teaching assistant recorded and organized questions submitted by the students. This important “trick of the trade” made communication easier between students and instructor, and helped the instructor to make optimally efficient use of her time.

As to the nature of discussions that took place throughout the course, two points were observed:

◇◇ As a result, students received feedback before submitting their work, and hence were eager to share their solutions on-line, in order to make their work as perfect as possible.

◇◇ The asynchronicity of time changed the character of group discussions, allowing the students to reflect more upon the comments of their teammates, as well as upon their own solutions, before responding to the whole group. In some cases, this resulted in displays of some profound mathematical thinking by the students.

◇◇ The written (as opposed to oral) character of discussions forced the students, through time, to be clearer and more concise in their expressions and ideas when they were communicating mathematics.

One disadvantage-turned-advantage was that there was no single compatible means with which to communicate mathematics over the internet. Students had to send their solutions as text files most of the time. In some cases this led to creative methods of communicating mathematics, sometimes forcing a different way of thinking about their solutions.

A research study was concurrently conducted, pertaining to the learning of linear algebra in a distance education context. Data for this study was collected in the form of videos of satellite sessions, records of on-line group discussions, student homework and projects, and exams. Sierpinska [**AS**] gave a characterization of students' theoretical thinking in linear algebra, identifying the following as features of practical, as opposed to theoretical thinking, based on a Vygotskian framework:

◇ transparency of language;

◇ lack of sensitivity to the systemic character of scientific knowledge;

◇ thinking of mathematical concepts in terms of prototypical examples rather than definitions;

◇ reasoning based on the logic of action, and generalization from visual perception.

The data analysis suggested that the way this course was structured favored certain aspects of theoretical thinking, in a way forcing the students to leave "practical practices" behind. For example, the written nature of the discussions gave way to gradually enhanced communications skills in group members. They had to be precise and concise with their explanations, otherwise their teammates did not understand their arguments. Hence, the students were able to display a language that was not transparent.

Also, Because of the asynchronous nature of the discussions, the students' reactions were not spontaneous; instead, they were posed after a period of reflection, which in turn gave rise to a quite rich exchange of ideas.

7. Tutorial Exercises

The original lecture featured an exerpt from Boats' Linear Algebra [**JB**] as an example of a computer tutorial exercise; that exercise explored the concept of spanning a vector space. Due to time constraints, a second example pertaining to the *QR-Algorithm with Shifting* was omitted from the lecture, but it is included here. It is also taken from [**JB**], and is slightly edited for brevity.

EXERCISE: The *QR Algorithm* for approximating eigenvalues.

Input the following matrices into MATLAB.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

a. Type the command $[\mathsf{Q}, \mathsf{R}] = \mathsf{qr}(\mathsf{A})$ to find a QR-decomposition for A. Then define the matrix A_1 with the command $\mathsf{A1} = \mathsf{R} * \mathsf{Q}$. Explain why A and A_1 must have the same eigenvectors.

b. We define the *QR Algorithm* as follows: for $i \geq 1$, construct the QR-decomposition of A_i, so that $A_i = Q_i R_i$, and then define $A_{i+1} = R_i Q_i$. Under suitable conditions (which we will discuss later), as i increases, the diagonal terms of the A_i's will converge to the eigenvalues of A. Use MATLAB to generate A_2 through A_5, and form a hypothesis about the eigenvalues of A. Test your hypothesis with the command $\mathsf{eig}(\mathsf{A})$.

c. To make the diagonal components of the A_i's converge to the eigenvalues of A faster, we can use *shifting*. We pick a real number t which we believe is "close" to an eigenvalue of A. Then at each step of the algorithm, we subtract tI from A_i, perform the same operations as before, and then add tI back at the end. For the matrix A and the value $t = 0.9$, the code for the first step of this shifting algortihm would be the following.

```
t=0.9;
S=A-t*eye(2);
[Q,R]=qr(S);
S1=R*Q;
A1=S1+t*eye(2)
```

Use this shifting technique on A with $t = 0.9$ and comment on what you observe.

d. Apply the *QR Algorithm* with MATLAB to approximate the eigenvalues of B. Then use a shift of $t = 3$ in applying the *QR Algorithm* to the matrix C. Then try again with $t = 3.5$ and $t = 4$. Which value of t is best? Why?

e. The *QR Algorithm* does not succeed for every square matrix. Run the algorithm for the matrix C and observe the results. Then find its eigenvalues by hand.

f. A sufficient condition for convergence of the algorithm is that the eigenvalues of the matrix are real and can be ordered $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n| \geq 0$, i.e. no two eigenvalues can have the same absolute value. Knowing this, use a shift to help the *QR Algorithm* find the eigenvalues of D. (For a detailed discussion of the convergence of the *QR Algorithm*, please refer to Sewell [**SE**], and for a discussion on the shifting technique, see Wilkinson and Reinich [**WR**])

The steps of the exercise lead the student through a guided tour of the theory behind the *QR Algorithm*. The students begin in part **a** by discovering that A and A_1 are similar, giving them the same eigenvalues. They observe the slow convergence of the diagonals of the A_i's in part **b**, and in part **c** they observe a faster convergence through shifting.

They are expected to give verbal explanations and form hypotheses along the way, and particularly so in part **d**, where they make the key observation that the better shifts are those which are "close" to an eigenvalue. Of course, the proof of why this is true is far beyond the scope of a linear algebra course, so a reference ([**WR**]) is given in lieu of a proof.

Part **e** takes responsibility for informing the students that the *QR Algorithm* without shifting can be insufficient. Finally, part **f** performs the most important duty of the tutorial exercise – it gives the students an opportunity to apply what they have learned.

The exemplary tutorial exercise guides the students through an explorative journey, with (hopefully) just enough guidance to prevent them from being led astray. The computer tutorial allows this exploration to proceed without the distraction of arithmetic. The students are able to focus instead on crucial theory or application, making key observations which will foster the appropriate mindsets for future concepts.

This is opposed to the many common misuses of instructional technology. Among these misuses are text-provided programs which solve the problem for the students (robbing them of the very process through which they learn), program-writing assignments (which improve the students programming abilities, but not their mathematics abilities), or pointless computer drillwork. The key to sucessful use of instructional technology, it seems, is to allow the technology to do *only* those things the students would not benefit from doing themselves.

The author encourages feedback and welcomes collaboration and discussion, and may be professionally contacted in the following ways:

Dr. Jeffery J. Boats
University of Detroit Mercy
Department of Mathematics and Computer Science
4001 W. McNichols Road, P. O. Box 19900
Detroit, MI 48219 (U.S.A.)

e-mail:	boatsjj@udmercy.edu
FAX:	(313) 993-1166
Phone:	(313) 993-1060

Resources

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[AE] http://wwwallexperts.com/browse.asp?Meta=8

[AO] portions of this section were taken with permission from abstracts of lectures given by Asuman Oktaç

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◇◇ All internet addresses valid August 12, 2000.