Geometric Conjectures: The Importance of Counterexamples

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To develop students' reasoning skills, the NCTM *Principles and Standards* recommends making generalizations and evaluating conjectures. In particular, it is emphasized that middle school mathematics students should be engaged in activities involving pattern recognition as a means of formulating such conjectures.

Explorations in geometry can be especially fruitful in providing opportunities for students to engage in discovery activities that help them develop inductive reasoning skills in making generalizations. Classroom discourse in deciding the truthfulness of their conjectures helps students develop deductive reasoning, as they strive to formulate valid arguments to convince others of the truth of their generalizations. In many of these geometry discovery lessons, teachers can help students acquire the reasoning strategy of the counterexample with timely provision of false conjectures for which counterexamples can be found.

For example, consider the sequence, borrowed from a primary school lesson: **1, 3, 6, 10, 15, 21, ??, ??, 45, ??, ...**

When students are asked to find what numbers go in place of the question marks, they look for a pattern. Once the pattern is found, a child typically explains it as "I added one more each time than I did before." The hypothesis is tested and found to work.

The students have determined a rule that *always worked.* It was *always* true. However, there are two other possible results from testing a conjecture that are often undervalued in the classroom: *sometimes* true (meaning the conditions were not specific enough, leading to ambiguous conclusions) and *never* true. More than any other area of mathematics, the geometry curriculum abounds with opportunities for students to investigate whether a conjecture is always, sometimes, or never true.

THE VALUE OF 'SOMETIMES'

Friedlander and Herschkowitz (1997) found that "many investigations that involve the processes of generalizing and justifying patterns at the level of beginning algebra require a student to follow a certain sequence of steps:

- Producing additional examples of the same kind;
- Employing the evolving pattern in some given situation;
- Generalizing the pattern;
- Justifying conclusions."

We have found the same process apparent in geometry investigations, with a notable difference. The topics discussed in geometry are easily visualized. The fact that students can draw representations of a geometry conjecture makes it easier for them to point out reasons for their arguments. This helps them communicate ideas orally to their classmates and the teacher – a nice stepping-stone on the path to convincing others with written arguments.

Specifically, the third step mentioned above, "generalizing the pattern," is the step where much interesting student interaction takes place. As students speculate as to what will happen under a specific change in a geometric figure, they make many conjectures about geometric relationships that turn out *not* to be true. They begin to use, and realize the value of, counterexamples as they investigate why something that seems as though it might be true, is not.

Statements that are sometimes true lead to very fruitful classroom discussions in which students not only make and justify mathematical arguments, but also are encouraged to reflect on their own reasoning. In these cases, a class will often split into two sides, each side finding examples to support their claim. In resolving the disagreement, students learn the strategy of the counterexample and come to understand that just one counterexample is sufficient to refute a universal statement. Students also begin to distinguish between the givens and the conclusion of a conjecture, a difficult distinction for many students (Galbraith, 1995).

One of the objectives of classroom discourse in developing reasoning skills is to help students avoid an incorrect expectation that their hunches about what at first seems reasonable will always generalize. In discovering the falsity of "wrong ideas," students can be motivated to search for regularities in their data and to use inductive reasoning to make generalizations that are true. They also start to become appropriately cautious in making inferences and accepting generalizations without careful examination. These are valuable metacognitive skills that will serve students in everyday life, as well as in the study of mathematics.

IN THE CLASSROOM

With the permission of teacher Mark Fratella and Our Lady Queen of Martyrs middle school in Birmingham, Michigan, we carried out several investigations with some of his seventh grade mathematics students. In these investigations, students explored the validity of several geometric conjectures related to squares.

We didn't mention at the start that conjectures can be "sometimes" true – instead, we let them discover this and describe it in their own words. The students worked together in groups of three, taking turns in the role of recorder, to encourage intra-group discussions.

Sample Investigation. *Begin with a square. Suppose you double the length of each side. What is the relationship between the perimeter of the first square and the perimeter of the second?*

What do you think will happen? Make a conjecture. Then try several different squares, each time making a new square using the method above. Record your results in a table. What do you notice?

When asked what they thought would happen, students called out that "the perimeter would get bigger," and "it will double." We carried out this investigation, having the students fill in their tables as we filled out a duplicate on an overhead projector.

The students all agreed that the conjecture seemed to be always true. We encouraged students to use the last line of their table to explore the general case of an initial square of side length n, as a way of employing concepts from algebraic reasoning to enhance their geometric understanding.

Investigation 1. *Begin with a square. Suppose you increase the length of the top and bottom by one unit each and then decrease the left and right sides by one unit each. What will be the relationship between the area of the square and the area of the new rectangle?*

A flurry of student responses came quickly. Some students thought the area would increase, because "the rectangle is taller." Others argued for decrease, because "it is skinnier." Most, however, believed there would be no change, because "you subtracted the same as you added."

In reasoning about the conditions, several students had fixed on the lengthened side as the salient feature of the resultant rectangle, and guessed that the area would increase. Others had fixed on the smaller side, and guessed the area would decrease. The majority had fixed on the amount by which the side lengths had increased or decreased.

The proposal of several hypotheses, including the majority hypothesis that there would be no change, stimulated discussion both within groups and between neighboring groups, inspiring a genuine interest in discovering whose ideas were correct.

Majority Conjecture 1: *A square has the same area as the area of a rectangle with short sides one unit smaller and long sides one unit larger than the sides of the square.*

Students worked together to fill out a table with their data (see **fig. 1a**,) and in the ensuing discussion, it was quickly resolved among them that the conjecture proposed by the class majority was, in fact, never true. A visual model (see **fig. 1b**) completely satisfied the students that the area of the rectangle would be one square unit less than that of the original square. They found it more convincing than the algebraic proof, in which a square of side length n is considered, as most of them had not yet studied the multiplication of binomials.

Investigation 2: *Begin with a square. Suppose you double the length of each side. What is the relationship between the area of the first square and the area of the second?*

The class again called out their first impulses, which were divided between two conjectures. Most students thought that the area would also be doubled, while a few thought that it would be quadrupled. We let the students investigate (see **fig. 2a**), this time designating a new recorder for each group.

Majority Conjecture 2: *If the sides of a square are doubled, then the area of the square is doubled.*

It wasn't long before the students, by examining several cases and filling out a table (see **fig. 2a**), found that the majority conjecture wasn't true for any of the squares they studied. Many students were genuinely surprised by the pattern they observed in the table, but despite their surprise, they all agreed by the end that the area would always quadruple.

The students agreed that the either the visual model or its corresponding algebraic proof (see **fig. 2b**) would be sufficient to convince others that the new square would have four times the area of the original.

Classroom discourse of this kind can help students begin to recognize a complete, valid mathematical argument. By having to explain their thinking to others, students also learn how to formulate their arguments with sufficient clarity to convince others.

Evaluating never true statements should be an integral part of a child's mathematical experience. We believe, however, that it is the third type of statement, "sometimes true," that is the most important for developing reasoning skills.

Investigation 3: *Begin with a quadrilateral. Find the midpoints of two opposite sides. Draw the mid-line, a line connecting the opposite midpoints. What is the relationship of the mid-line to the other two sides?*

On the overhead projector, we demonstrated the construction of a mid-line using a square. The students noticed (among other things) that it was "parallel with the sides it doesn't touch." They began drawing quadrilaterals to investigate their final conjecture of the day (see **fig. 3a**).

Conjecture 3: *A line joining the midpoints of two opposite sides of a quadrilateral is parallel to the other two sides.*

This is an intelligent conclusion based on the square, or on the other figures shown in **fig 3b**. However, those figures have an extra condition placed upon them – they are all quadrilaterals with at least one pair of parallel sides. Although the conjecture is true for some quadrilaterals, it cannot be generalized to all quadrilaterals, as shown by the counterexample in **fig. 3c**.

By not thinking from the onset to draw quadrilaterals other than the standard square, rectangle, and parallelogram, some students made a false generalization. But after only a few minutes of drawing their own quadrilaterals, nearly the entire class was clamoring that the conjecture was not always true. "Sometimes it's true," they said, or "it depends." Another pointed out that "it's not true if you draw an odd shape."

"But sometimes it is true," another pointed out, drawing general agreement.

We stepped in at this point, and suggested that the class try to sort the quadrilaterals into two sets: ones for which the conjecture was true, and ones for which it was not. There was considerable debate among the students, for a while, regarding trapezoids, since one must consider both midlines in order to notice that the conjecture can fail (see **fig. 3d**).

After some discussion, there was general agreement that the statement is true for squares, rectangles, and parallelograms, and sometimes trapezoids, but seemed to be false for all other quadrilaterals.

Students had difficulty attempting to describe what properties of some quadrilaterals made the statement true when for all other quadrilaterals the statement was false.

> "The odd shapes aren't usually parallel."
>
> "Not all the quadrilaterals have even sides."
>
> "Some of the shapes have slanted sides."
>
> "It depends upon if the sides are equal in length and if their distances away are equal."
>
> "The shapes have to be lines that are parallel."
>
> "The sides have to be equal."
>
> " The hypothesis is usually what happens unless the shape doesn't have the same shape of the opposite side."

Gradually, the students began to remember that squares and rectangles are also parallelograms. They decided that having opposite sides parallel is the condition that makes the statement true. They also noticed that the bases of trapezoids are parallel, but the legs are not. For this reason, they concluded that the statement is true for only one mid-line of a trapezoid –the line connecting the midpoints of the two legs.

FURTHER EXPLORATIONS

As mathematicians will be quick to point out, counterexamples are often difficult to find. They often appear only after a long and arduous search. Asking children to validate mostly always true and never true conjectures can result in a false sense of having examined all the cases. The sometimes-true statements are needed to help children dig deeper, gather evidence, test conjectures, and look for irregularities. When children are engaged in thinking and reasoning about all three types of conjectures, teachers can help children become appropriately cautious in making inferences and in examining all the conditions.

We conclude by providing a list of more classroom investigations of propositions, most of which are sometimes true.

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More Classroom Investigations

Answer: **S** = Sometimes True, **A** = Always True, **N** = Never True

__A__ an obtuse triangle has 2 acute angles

__S__ a rectangle has perpendicular diagonals

__S__ a parallelogram has congruent diagonals

__S__ a rhombus has 2 acute angles

__A__ a square has perpendicular diagonals

__S__ an isosceles triangle is equilateral

__N__ the bisector of an obtuse angle divides the angle into two smaller obtuse angles

__S__ an octagon has 8 congruent sides

__S__ a parallelogram has perpendicular diagonals

__S__ a diagonal of a hexagon is a line of symmetry

__S__ the diagonals of a rectangle intersect to form 4 congruent triangles

__A__ the diagonals of a rhombus intersect to form 4 congruent triangles

__N__ if the radius of a circle is doubled, then the area of the circle is doubled

__S__ a rhombus has congruent diagonals

__S__ a median of a triangle forms 2 congruent triangles

__S__ the altitude of a triangle forms congruent triangles

__A__ the mid-line of a parallelogram is parallel to one side of the parallelogram

__N__ vertical angles have a common vertex and a common side

__A__ two consecutive angles of a parallelogram are supplementary

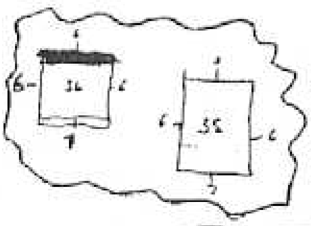
__S__ a bisector of a line segment is perpendicular to the segment

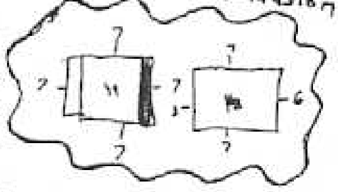
__S__ if all the angles of one figure are congruent to the corresponding angles of a second figure, then the figures are similar

__N__ the diagonal of a pentagon is a line of symmetry

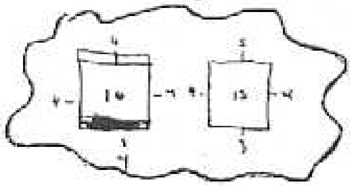
Hypotesis: The area of the square will stay the sum

Side of square	Area of square	Width of rectangle	Length of rectangle	Area of rectangle
6	36	5	7	35
7	49	6	8	48
4	16	3	5	15
5	25	4	6	24
3	9	2	4	8
9	81	8	10	80





Conclusion: Every tim we did this experiment the area of the square decreased by 1



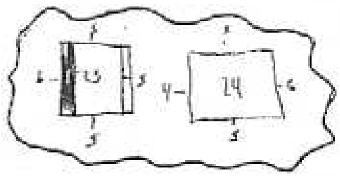
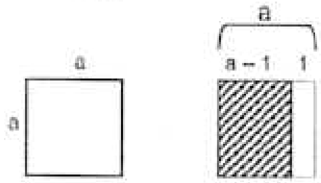


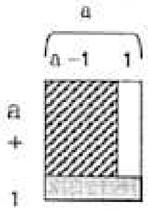
Figure 1a

Conjecture 1: A square has an area that is one square unit more than the area of a rectangle with short sides one unit smaller and long sides one unit larger than the sides of the square.

Area Proof:



Rearrange the regions:



The resultant rectangle has an area one square unit less than the original square.

Algebraic Proof:

$$
\begin{aligned}
(a + 1)(a - 1) &= a(a - 1) + 1(a - 1) \\
&= a^2 - a + a - 1 \\
&= a^2 - 1
\end{aligned}
$$

Figure 1b

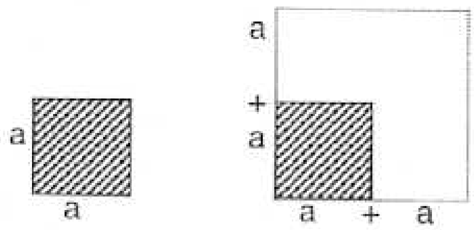
~ Math ~

Side of square	Area of square	Doubled side	Area of new square
6	36	12	144
10	100	20	
2	4	4	16
5	25	10	100
8	64	16	
3	9	6	36
9	81	18	

Hypothesis – I think it will coudruply because the numbers always turn out bigger.

Figure 2a

Visual Model: What happens to the area of a square when each side is doubled in length?



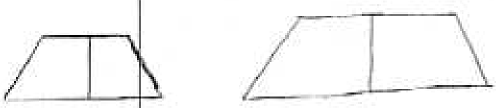
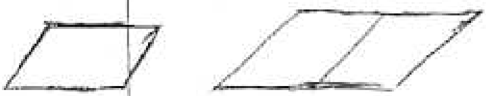
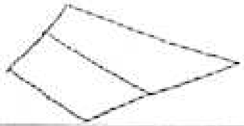
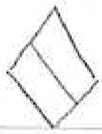
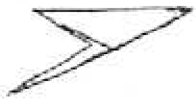
Area $= a \times a$
$= a^2$

Area $= (2a)(2a)$
$= 4a^2$
$= 4(\text{area of original square})$

Figure 2b

Ellie OK

Conjecture: The midsection of a quadrilateral is parallel to the sides it doe touch.

Sketch of quadrilateral	True	False
		✓
	✓	
		✓
	✓	
		✓

The Conjecture is false. It depends on if the quadrilateral's lines are all parallel or not.

Figure 3a

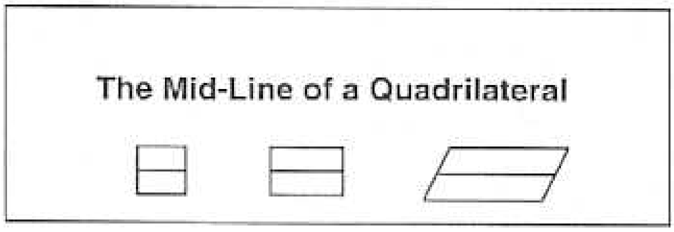
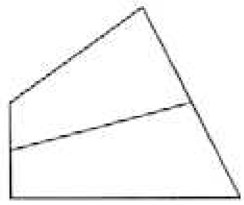


Figure 3b

A Counterexample



The mid-line is not parallel to the
remaining two sides

Figure 3c

The mid-line is parallel to the remaining two sides.

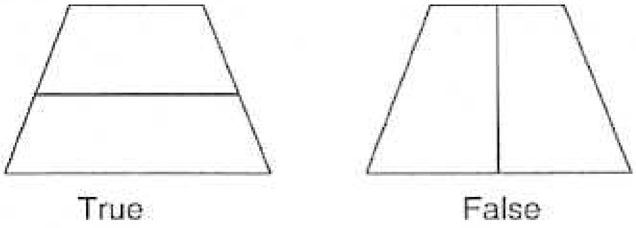


Figure 3d