FINDING DISJOINT PATHS IN THE NOVA GRAPH

JEFFE BOATS, LAZAROS KIKAS, JOHN OLESIK

Abstract. For the purpose of large scale computing, we are interested in linking computers into large interconnection networks. In order for these networks to be useful, the underlying graph must possess desirable properties such as a large number of vertices, high connectivity and small diameter. In this paper, we are interested in the Nova graph as an interconnection network, and the $k$-Disjoint Path Problem. In 2007, Boats, Kikas and Oleksić showed that the Nova graph $A_n^+4$ has the 3-Disjoint Path Property. In this paper, we extend the result to the general Nova graph $A_n^+$, and show that this class of interconnection networks has the $(n-1)$-Disjoint Path Property for $n \geq 4$. We discuss the significance of this result in comparison to other interconnection networks and close with remarks on possible future research directions.

Keywords: Interconnection networks, graphs, vertex disjoint paths

1. Introduction

The study of large interconnection networks has attracted a great deal of research over the last few years. In order for these networks to be useful, the underlying graph structure should have properties such as vertex symmetry, high connectivity, low diameter, and a large number of vertices (see [1, 2]).

In this paper we study the following problem: Given $k$ pairs of distinct nodes $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ in a graph, can these pairs be connected with $k$ disjoint paths? This problem is called the $k$-Disjoint Path Problem, and has generated much research. If for a graph $G$ we can do this for any selection of $k$ pairs of distinct nodes, then $G$ is said to have the $k$-Disjoint Path Property.

Much has been studied about this problem. It has been shown for $k \geq 3$ the problem of finding $k$ disjoint paths is $NP$-hard. Watkins in [9] showed that if a graph $G$ has the $k$-Disjoint Path Property, then it must be $(2k-1)$-connected. Past work done by Cheng and Lipman shows that the split star graph, $S_n^2$, has the $(n - 1)$-Disjoint Path Property [3]. In 2005, Cheng, Kikas, and Kruk showed that the alternating group graph $AG_n$ has the $(n - 2)$-Disjoint Path Property [4, 8].

Both of the above proofs, for the split star and the alternating group, were existence proofs, not providing an algorithm for the construction
of these paths. In 2006, Boats, Kikas, and Oleksik developed an algebraic/geometric algorithm that took advantage of the hierarchal structure of $AG_5$ to demonstrate the construction of the 3-disjoint paths [5].

In 2007, Boats, Kikas and Oleksik developed a generalized algebraic approach for finding disjoint paths in the alternating group graph $AG_n$ [7]. In a separate paper in 2007, Boats, Kikas and Oleksik introduced the Nova graph as a new interconnection network, and showed that the Nova graph $\mathcal{A}_4^+$ has the 3-Disjoint Path Property [6]. In this paper, we generalize our result to $\mathcal{A}_n^+$ and show that this class of interconnection networks has the $(n - 1)$-Disjoint Path Property for $n \geq 4$.

2. COMPOSITION AND CONNECTIVITY OF THE NOVA GRAPH

The Nova graph is formed from the alternating group graph, $AG_n$, $n \geq 4$, whose vertices are the even permutations of the first $n$ natural numbers. Two vertices are connected by an edge if and only if one of the vertices’ corresponding permutations can be obtained from the other by means of a permutation of the form $(12j)$, with $j \in \{3, 4, \ldots, n\}$. This makes $AG_n$ a Cayley graph, with these $n - 2$ edge permutations as generators. The degree of each vertex is $2n - 4$, since there will be an adjacent vertex for each of the $n - 2$ generators, and also each of the generators’ inverses, $(1/j)$.

The Nova Graph, $\mathcal{A}_n^+$, is formed from $AG_n$ by adding edges defined by the permutation $J = (12)(34)$. It was shown in [6] that this decreased the diameter of $\mathcal{A}_4^+$ to two, enabling it to have the 3-disjoint path property. The Nova Graph is also a Cayley graph, with $n - 1$ generators. The degree of each vertex is $2n - 3$; that includes the $2n - 4$ adjacent vertices from before, and one more adjacent vertex connected by a “$J$” edge. Figure 1 illustrates $\mathcal{A}_4^+$. 

![Figure 1. The Nova Graph $\mathcal{A}_4^+$](image-url)
Like $AG_n$, $A_n^+$ has a hierarchical structure, consisting of $n$ “substars,” denoted $H_1, H_2, \ldots, H_n$, each with identical structure to $A_{n-1}^+$. The index $j$ in $H_j$ indicates the last digit in the even permutation associated with the vertex. For this reason, every vertex in the graph is adjacent to $2n - 5$ vertices within the same substar, and one more vertex in each of 2 other substars. Those two vertices are adjacent through $(12n)$ and $(1n2)$ edges, and they are also adjacent to each other, thus forming a 3-cycle. These $(12n)$ and $(1n2)$ edges are called “conduits.”

There will be times when we wish to begin a path from $s_i$ to $t_i$ by immediately leaving $s_i$’s substar via a conduit edge; we will refer to this as “quickgating.” The vertex $v_1$ can “quickgate” to vertices $v_2 \in H_y$ and $v_3 \in H_z$, where $y$ and $z$ are the first two integers in the even permutation corresponding to $v_1$. The only 3-cycles which involve more than one substar necessarily involve three different substars, such as the $\{v_1, v_2, v_3\}$ example above, so it follows that two vertices in the same substar cannot be adjacent to the same vertex in another substar. This is useful to know, as it means whenever we choose to quickgate multiple vertices from the same substar, all of them will necessarily go to different places.

Between any 2 substars of $A_n^+$ are found $K = (n - 2)!$ conduits. This is an important observation, since we will want to use conduits whenever routing a path for any $(s_i, t_i)$ pair not contained within a single substar. If neither $s_i \in H_y$ nor $t_i \in H_z$ lies on a conduit connecting their substars, we may attempt to route $s_i$ to a vertex $\sigma_i \in H_y$, and $t_i$ to $\tau_i \in H_z$, where $\sigma_i$ and $\tau_i$ occupy opposite ends of a conduit. This is called “surrogate routing,” and $\sigma_i$ and $\tau_i$ are the “surrogates” of $s_i$ and $t_i$ respectively.

### 3. Preliminary Considerations

Suppose $M_{ij}$ is the number of mated pairs which must be routed between $H_i$ and $H_j$. Before routing them, we must consider not only the value of $K$, but also $B_{ij}$, the number of conduit vertices occupied by elements of $S \cup T$ not to be so routed. The $B_{ij}$ vertices are “blocks,”– they render their conduits useless since the $M_{ij}$ paths cannot be routed through them.

We must also consider how the vertices of $S \cup T$ are distributed among the substars. The following definition effectively categorizes the possible arrangements.

**Definition** For the vertices $S \cup T \subset A_n^+$, their phi-count is the ordered $n$-tuple $(\phi_1, \phi_2, \ldots, \phi_n)$, where $\phi_\alpha = |(S \cup T) \cap H_\alpha|$. We denote by $\phi_{\max}$ the largest number in a phi-count:

$$\phi_{\max} = \max_\alpha |(S \cup T) \cap H_\alpha|.$$

There are two lemmas which we will need during the proof of the main result. The first establishes a condition on $\phi_{\max}$ which will guarantee the
existence of sufficient disjoint paths, while the second will help prove a
difficult special case.

**Lemma 1**  
In $A_{n+1}^+$, with $n \geq 4$, if $\phi_{\text{max}} \leq n - 1$, then the vertices of $S \cup T$ can be connected with $n$ disjoint paths.  

**Proof:** Let $K$ be the number of conduits between substars in $A_{n+1}^+$: for $n \geq 4$, we have $K = (n - 1)! \geq 2(n - 1)$.  
Let $M_{i,j}$ be the number of mated pairs to be routed between substars $H_i$ and $H_j$, and $B_{i,j}$ be the number of blocked conduits between those substars. Every vertex of $S \cup T$ could potentially block a conduit, but for every mated pair to be routed between the substars, at least one conduit (if not two) are still usable. Thus: $B_{i,j} \leq \phi_i + \phi_j - M_{i,j}$.  
For routing the $M_{i,j}$ pairs via conduits to be impossible, it would take:

$$
M_{i,j} > K - B_{i,j} \\
\geq K - (\phi_i + \phi_j - M_{i,j}) \\
= M_{i,j} + K - \phi_i - \phi_j \\
\Rightarrow \phi_i + \phi_j > K \geq 2(n - 1)
$$

Thus, $\phi_{\text{max}} \leq n - 1$ is sufficient to guarantee there will be enough unblocked conduits for routing.  
Once suitable open conduits are found for all inter-substar $S \cup T$ pairs, we can always use these conduits for surrogate routing. Each substar of $A_{n+1}^+$ has the $(n - 1)$-disjoint path property, and since $\phi_{\text{max}} \leq n - 1$, it will always be possible to route vertices of $S \cup T$ to surrogates.  

This lemma lends itself to a general strategy for proving $A_{n+1}^+$ has the $n$-disjoint path property. If the vertices of $S \cup T$ are distributed such that $\phi_{\text{max}} \leq n - 1$, or if we can route some or all of the $S \cup T$ vertices to new positions in such a way that the new $\phi_{\text{max}} \leq n - 1$, then the result follows. Indeed, explaining how this can be done for various values of $\phi_{\text{max}}$ will constitute the bulk of the induction step of the proof.

**Lemma 2:** For $n \geq 4$, if $A_{n+1}$ has substars $H_y$ and $H_z$ such that $S \subset H_y$ and $T \subset H_z$, then it is always possible to find two mated $(s_i, t_i)$ pairs for which both vertices can quickgate without being blocked.  

**Proof:** Since each vertex of $S \cup T$ can quickgate to a vertex in two different substars, one of the two substars must be empty. It follows that any given mated pair, $s_i$ and $t_i$, can each quickgate without being blocked. This is because $s_i$ has at least one open vertex to which it may quickgate, and the only way $t_1$ could have both its quickgates occupied is if $s_1$ was routed to one of them, but $t_1$ cannot be blocked by its mate.
Quickgate \(s_1\) and \(t_1\); we name their destinations \(w_1\) and \(w_2\) respectively. It is impossible for \(s_1\) to block a quickgate destination for another vertex in \(S\), because two vertices in the same substar can’t quickgate to the same destination. The vertex \(s_1\) could block one of the \(t_i\)’s, but no more than one, since \(w_1\) is adjacent to only two vertices from other substars, and \(s_1\) is already one of them. Similarly, \(t_1\) is also capable of blocking only one of the \(s_i\)’s.

Thus the mated pair \((s_1, t_1)\) is capable of blocking at most two other mated pairs. Since \(n \geq 4\), there are at least three other mated pairs, and therefore a second pair will always be able to quickgate. □

4. Proof of the Proposition

**Proposition** For \(n \geq 4\), \(A_n^+\) has the \((n - 1)\)-disjoint path property.

This will be a proof by induction, using \(n = 3\) (i.e. \(A_4^+\)) as the base case. It has already been proven by Boats, Kikas, and Oleksik in [6] that \(A_4^+\) has the 3-disjoint path property, so only the induction step remains; its proof will take up the remainder of this section.

For the induction step, we assume \(n \geq 4\) and that \(A_n\) has the \((n - 1)\)-disjoint path property. It follows that the substars \(H_i, i \in \{1, 2, \ldots, n+1\}\), which comprise \(A_{n+1}^+\) each have the \((n - 1)\)-disjoint path property. The object will be to show that \(A_{n+1}^+\) has the \(n\)-disjoint path property.

Let the \(S \cup T\) configuration be given, and consider the value of \(\phi_{\text{max}}\).

The \(\phi_{\text{max}} \leq n - 1\) cases: By Lemma 1, \(n\) disjoint paths can be found.

The \(\phi_{\text{max}} = 2n\) case: We can quickgate every \(S \cup T\) vertex to one of two neighbors with no fear of blocking, and if the new \(\phi_{\text{max}}^* \leq n - 1\), then the \(n\) disjoint paths can be completed by Lemma 1.

Suppose \(\phi_{\text{max}}^* \leq n - 1\) isn’t possible. Since any two \(S \cup T\) vertices can be quickgated to different substars, it can only be that all or all but one of the \(S \cup T\) vertices quickgate to the same two substars. In either case, the quickgate destinations can be selected so that \(n - 1\) mated \((s_i, t_i)\) pairs are sent to some substar \(H^*\), and the other two \(S \cup T\) vertices are sent anywhere else. Since \(H^*\) has the \((n - 1)\)-disjoint path property, those \(n - 1\) disjoint paths can be completed; it is then a trivial matter to complete the last path through surrogate routing.

The \(\phi_{\text{max}} = 2n - 1\) case: Without loss of generality, say \(s_1 \in H_2\), which means \(\{S \cup T\} \setminus s_1 \subset H_1\). We then quickgate some \(S \cup T\) vertex in \(H_1\) to a substar other than \(H_2\), say \(H_3\); if \(s_1\) blocks a quickgate destination for some vertex in \(H_1\) other than \(t_1\), we necessarily choose this vertex to quickgate to \(H_3\).

If \(\phi_{\text{max}} \leq n - 1\) is possible we can apply Lemma 1, so assume not. It follows that all or all but one of the remaining \(S \cup T\) vertices in \(H_2\) quickgate
to the same two substars, and that at least one of those substars must be $H_1$ or $H_3$. As in the previous case, we can choose quickgate destinations so that $n - 1$ mated pairs end up in the same substar ($H_1$ or $H_3$), with the other two vertices ending up elsewhere, and the result follows by the same argument as in the previous case.

The $n \leq \phi_{\text{max}} \leq 2n - 2$ cases: We categorize these cases as $\phi_{\text{max}} = 2n - x$, where $x$ is the number of vertices not in $H_1$. We will wish to connect some mated pairs within $H_1$, and route unmated $S \cup T$ vertices to conduits in order to send them to their mates’ substars.

Since $H_1$ has the $(n - 1)$-disjoint path property, we can route all we want provided we use no more than $2(n - 1)$ total surrogate and $S \cup T$ vertices; hence the number of unmated vertices we can route to any conduits we wish is:

$$2n - 2 - \phi_{\text{max}} = 2n - 2 - (2n - x) = x - 2$$

Define $\psi$ to be the number of mated pairs in $H_1$; clearly $\psi \geq n - x$. Since $H_1$ has the $(n - 1)$-disjoint path property, we can easily connect all $\psi$ mated pairs with disjoint paths. After routing these mates together, the number of unmated vertices left is $\phi_{\text{max}} - 2\psi$.

Consider the subcase where not every $S \cup T$ pair has a representative in $H_1$; i.e. there exists a mated $(s_1, t_1)$ pair with neither $s_1$ nor $t_1$ in $H_1$. It follows that $\psi \geq n - x + 1$. This means the number of unmated vertices in $H_1$ is:

$$\phi_{\text{max}} - 2\psi = 2n - x - 2\psi \leq 2n - x - 2(n - x + 1) = x - 2.$$ 

Thus every $S \cup T$ vertex can either be routed to its mate in $H_1$, or routed to an open conduit from $H_1$ to the substar hosting its mate; the $n$ disjoint paths are then easily completed.

All remaining subcases involve $S \cup T$ configurations for which every mated pair has a representative in $H_1$, and since switching the $s_i$ and $t_i$ designations doesn’t change the problem, we can categorize these subcases, without loss of generality, as $S \subset H_1$. The number of unmated vertices in $H_1$ is:

$$\phi_{\text{max}} - 2\psi = 2n - x - 2\psi = 2n - x - 2(n - x) = x,$$

which means we can connect mated pairs in $H_1$ and direct all but two of the unmated $S \cup T$ vertices to appropriate conduits.

Therefore, the existence of $n$ disjoint paths will be proven if we can demonstrate that it is always possible to quickgate two unmated $S \cup T$ vertices, avoiding blocks, and that doing so will create a new configuration.
There are only two unmated \( \phi \)'s eligible for quickgating, with the \( x \) unmated \( t \)'s as potential blocks. Two \( s \)'s cannot quickgate to the same vertex, which implies that they cannot block each other, and also that it takes two \( t \) blocks to prevent an \( s \) from quickgating. Thus \( Q \), the number of \( s \)'s which can quickgate without blockage, is:

\[
Q \geq x - \left\lfloor \frac{x}{2} \right\rfloor.
\]

The \( x = 2 \) subcase: There are only two unmated \( s \)'s in \( H_1 \), without loss of generality say \( s_1 \) and \( s_2 \). The vertex \( s_1 \) cannot be blocked from quickgating, as this would require two blocks, and the only potential block is \( t_2 \) since \( t_1 \) is \( s_1 \)'s mate and thus its destination, not a block. Similarly, \( t_2 \) cannot be blocked. If \( s_1 \) is sent to a different substar than \( t_1 \), and \( s_2 \) is sent to a different substar than \( t_2 \), then \( \phi_{\max}^* = 2 \leq n - 1 \) for \( \{A_{n+1}^+ \setminus H_1\} \), and the result follows from Lemma 1.

The \( 3 \leq x \leq n - 1 \) subcases: \( x \geq 3 \) implies that \( Q = x - \left\lfloor \frac{x}{2} \right\rfloor \geq 2 \), so it is possible to quickgate two or more unmated vertices from \( S \cup T \).

Since \( x \leq n - 1 \), we are guaranteed that the \( \phi \)-value is no more than \( n - 1 \) for any substar other than \( H_1 \). Suppose \( \phi_{\max}^* \leq n - 1 \) isn’t possible after two quickgates. This implies one or more blocks outside of \( H_1 \) which force the quickgating \( s \)'s to a substar \( H^* \) so that its \( \phi \)-count becomes \( n \) or more. This would require the number of blocks plus the number of \( t \)'s in \( H^* \) to be at least \( n \), which contradicts \( x \leq n - 1 \).

Hence \( \{A_{n+1}^+ \setminus H_1\} \) has \( \phi_{\max}^* \leq n - 1 \); the result follows from Lemma 1.

The \( x = n \) subcase: This implies \( S \subset H_1 \) and \( T \subset \{A_{n+1}^+ \setminus H_1\} \).

If the \( \phi \)-count is no more than \( n - 1 \) for any substar other than \( H_1 \), then \( \phi_{\max}^* \leq n - 1 \) is possible after two quickgates, by the same argument as in the previous subcase.

This leaves only the special subcase where \( T \) is contained entirely within one substar, without loss of generality say \( S \subset H_1 \) and \( T \subset H_2 \). By Lemma 2, we can quickgate two mated pairs, say \((s_1, t_1)\) and \((s_2, t_2)\), without blockage. We can then use surrogate routing to send every other vertex in \( S \cup T \) to any of the other \( n - 1 \) substars we choose.

If \( \{s_1, s_2, t_1, t_2\} \) all quickgate to the same substar, say \( H_3 \), then we can use surrogate routing to send every other vertex in \( S \) to \( H_4 \), and every other vertex in \( T \) to \( H_5 \) (for \( n \geq 4 \), there are at least 5 substars). The \((s_1, t_1)\) and \((s_2, t_2)\) paths can be easily completed within \( H_3 \), and the remaining paths can be completed by Lemma 1, since the new configuration in \( H_4 \cup H_5 \) has \( \phi_{\max}^* = n - 2 \leq n - 1 \).

If no more than 3 of \( \{s_1, s_2, t_1, t_2\} \) go to the same substar, then we choose surrogate routes for the rest of \( S \cup T \) which spread the vertices out as much
as possible over the substars of \( \{A_{n+1}^+ \setminus (H_1 \cup H_2)\} \). Spreading \( 2n \) vertices over these \( n - 1 \) substars guarantees that the resulting new \( \phi \)-count has:

\[
\phi_{\text{max}}^* = \min\left\{ 3, \left\lceil \frac{2n}{n - 1} \right\rceil \right\},
\]

which for \( n \geq 4 \) is guaranteed to be no more than 3. Thus \( \phi_{\text{max}}^* \leq n - 1 \), and the \( n \) disjoint paths can then be completed by Lemma 1.

As we have covered all possible \( \phi \)-counts, the proposition is proven. □

5. Performance Advantages of the Nova Graph

With the proposition proven, it is pertinent to consider the practicality of the result. Akers [2] first postulated numerous advantages to the use of symmetry groups for graph structures in interconnection networks. The alternating group graph \( AG_n \), the split-star graph \( S^2_n \), and now the Nova graph \( A^+_n \) are the most recently-studied graphs of this type. Consider the following table, summarizing the disjoint path performance and several characteristics of these graphs.

<table>
<thead>
<tr>
<th></th>
<th>( AG_n )</th>
<th>( S^2_n )</th>
<th>( A^+_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disjoint Paths</td>
<td>( n - 2 )</td>
<td>( n - 1 )</td>
<td>( n - 1 )</td>
</tr>
<tr>
<td>Vertices</td>
<td>( \frac{n!}{2} )</td>
<td>( n! )</td>
<td>( \frac{n!}{2} )</td>
</tr>
<tr>
<td>Edges</td>
<td>( (n - 2)n! )</td>
<td>( \frac{(2n - 2)!}{2}n! )</td>
<td>( \frac{(2n - 2)!}{n!}n! )</td>
</tr>
<tr>
<td>Diameter</td>
<td>( \left\lfloor \frac{3(n - 2)}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{3(n - 2)}{2} \right\rfloor + 1 )</td>
<td>( \left\lfloor \frac{3(n - 2)}{2} \right\rfloor ), ( n = 4 )</td>
</tr>
<tr>
<td>Vertex symmetry</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Edge Symmetry</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

The “\( J \)” edges used in constructing the \( A^+_n \) from \( AG_n \) prevent the Nova graph from being edge-symmetric, but this is a small sacrifice which yields big dividends. In guaranteeing as many disjoint paths as the split-star, the Nova graph uses half as many vertices and half as many edges. It does so with as small a diameter as the alternating group graph, and in the case of \( A^+_4 \) the diameter is even smaller.

6. Conclusion

In this paper we have proven that the Nova graph, \( A^+_n \), has the \( (n - 1) \)-disjoint path property. Our proof is purely an existence proof, that is, we have not provided an algorithm which generates the paths. However, we believe that the generalized algebraic algorithm for finding disjoint paths in \( AG_n \) [7] can be adopted to find the disjoint paths in any Cayley graph, including \( A^+_n \). This, combined with the performance advantages of the Nova graph, suggest it is desirable for use as an interconnection network.
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