UNIVERSITY OF DETROIT MERCY

Narrative for BOATS' Proposal: "Adaptation of Gaussian Elimination to Non-Fields Toward the Characterization of Group-Magic Graphs"

1. **Brief Abstract: Please describe your proposal in 100 words or less. Please remember to use language or define terms so that faculty members from different disciplines are able to understand your intent.**

Gaussian Elimination is the standard way of solving systems of linear equations. It only works in number sets which feature both multiplication and addition, like the real numbers do. Yet there are many important applications in which systems arise, but in contexts where only a single operation is permitted. Normal Gaussian Elimination won't work there; we propose a generalized version which can still extract meaningful results. This project will formalize this idea, put it into a working C++ code for easy implementation, and use the code in an experimental attempt to solve long-standing, open problems in Graph Theory.

2. **What is the goal and/or purpose of the research/scholarship?**

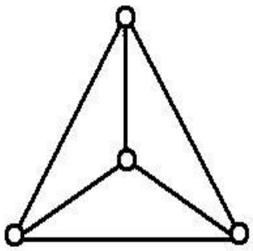
The goal is to explore open problems in graph theory, specifically group-magic labelings, by encoding an algorithm with generalizes Gaussian elimination, and then using it as a new and powerful tool for experimentation. I will attempt in the following paragraphs to explain our research goals in non-technical terms.

Gaussian elimination, as found in any textbook on Linear Algebra (including my own **[1]**), requires both addition and multiplication operations. While much of mathematics deals only with "fields," which are sets featuring both of these operations and all their convenient properties, there are many important applications in which we have only one operation which can be used.

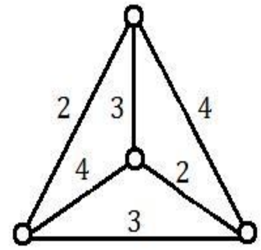
Definition: A group is closed set, usually of numbers, which has only one associative operation, an identity element (sometimes called the "zero"), and the existence of inverses with respect to that identity element.

We will develop a computational tool which will generalize Gaussian elimination to any group. The impetus for this investigation comes from the study of Group-Magic graphs.

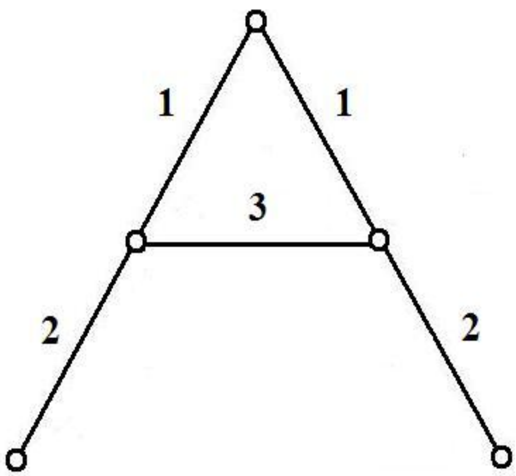
Definitions: A graph is a set of vertices (points), some of which are connected by edges (connecting curves). A graph is group-magic if there is at least one way to place a non-zero element as a label on each edge so that, at every vertex, the sum of all the edge labels is the same.



Example: The graph on left, shown without labels, is called K-4. It has four vertices and six edges. On the right is a magic-labeling for K-4, whose edges at each vertex sum to 9.



In magic labeling, each vertex represents a linear equation, and each edge is a variable, whose values are its possible non-zero labels. Determining whether a graph has a group-magic labeling is a matter of solving a system of linear equations. The catch is that in a group, rather than a field, standard Gaussian elimination doesn't work. The purpose of this project is to generalize the procedure, creating a tool to solve group-magic labeling problems, and in all likelihood many other open problems in graph theory in the future. One such open conjecture is illustrated below:



The graph on the left does not have a magic labeling for the groups Z_2 or Z_{2x2} , but it does for Z_4 .

One Z_4 -magic labeling is shown. The non-zero elements of Z_4 are { 1 , 2 , 3 }, and addition is performed modulo-4, meaning that two numbers are equivalent if they have the same remainder when divided by 4. (That's why it's OK for some vertex sums to be 2, while other vertices sum to 6)

It has been conjectured that any Z_{2x2} -magic graph must necessarily by Z_4 -magic, but this is unproven.

3. How does the research/scholarship relate to the mission of UDM?

This particular topic is very accessible students, even those of majors besides mathematics and software engineering, because it involves problem solving using modular arithmetic (not a difficult concept) and logic / critical thinking (challenging, but hopefully universal among UDM students). Some areas of mathematics require years of graduate study before one can engage in meaningful research, but graph theory is accessible to any good mathematics major, and top students from other disciplines.

I'm looking forward to working closely with two undergraduate students in research next Winter, and opening the project up to a wider group in the future. I see undergraduate research as a key component to student-centered teaching, as it teaches them not only the subject area, but the nature of professionalism and perseverance in science research – something they'll need in their future careers.

4. Please describe your methods. Include research design if relevant.

Mathematics is more like the physical sciences than many physical scientists realize. We conduct an enormous amount of experimentation, not with test tubes or circuitry, but with algorithms, logic, pattern recognition, and various computational techniques. This research project began shortly after my colleague Richard Low related some open problems to me over lunch at a graph theory conference, illustrating a few key points on his napkin. After some thought, I realized my expertise in a seemingly unrelated area, linear algebra, would enable me to construct a powerful algorithm which could solve his problems, and probably many others. That's how it begins, sometimes.

My research group will encode my algorithm in C++. This is the most important aspect of the research design, because computational speed is very important to this type of experimentation. While I can prove my algorithm will work, it would take hours to perform the algorithm by hand for any graph of significant size, whereas a computer would finish in seconds. To ensure proper encoding, I will work closely with research colleagues who have extensive programming experience. I will be asking for special laptops with quad-core processors to ensure shorter run times, and an external drive for storing enormous data sets; some of the algorithm outputs are too big to store in a standard laptop!

Once the code is operational, the true experimentation begins, when we run it on numerous graphs with many different group structures. We will verify a few previously-known results to test its efficacy. Specifically, we will begin by verifying a result from **[8]**. Theorem 1 of that paper states an interesting result for the group "Z3," which is the cyclical group of only 3 elements. Although this is a very simple group structure, there is very little known about magic-group labeling for it beyond said Theorem 1. Z3 also has the convenient feature it is a "field," which in layman's terms means any elimination algorithm is guaranteed to arrive at a solution. Verifying the theorem and then expanding on it will be a good way to validate our algorithm and the meaningfulness of its results.

At that point, there will be ample opportunity for undergraduates to join in our research. In selecting the two students who will help with our research, we will look for students with excellent programming skills and who have taken MTH4020, UDM's Linear Algebra course for scientists and engineers, as well as MTH4600, an introductory course in Graph Theory. They will sort through the immense data sets of algorithm output, helping us spot the patterns (which classes of graphs have group-magic labeling for which groups) that will guide our theoretical explorations.

We will input numerous graph structures in order, beginning with the smallest and simplest and building gradually upward, determining which ones have solutions, and look for patterns. We hope to completely categorize Z3-magic labelings (i.e. determine necessary and sufficient conditions under which any given graph would have a Z3-magic labeling), though even a partial categorization of Z3-magic labelings would be truly groundbreaking. Beyond that, we will explore other group structures, categorizing them as much as possible. We will begin with simple graph structures, such as those in **[10]**, **[11]**, and **[12]**, and also explore certain larger graph structures which appear throughout graph theory literature due to their usefulness in applications and/or the frequency with which they appear in other research literature; reference **[9]**, for example, generalizes magic-labeling results to products of graphs. There is literally no end to the number of directions we can head; we will allow early results to guide later pursuits.

5. What are the expected outcomes and how do you intend to measure them?

The first outcome is the composition of an algorithm which will generalize Gaussian elimination to groups. This outcome will be assessed as "successful" if I can construct a mathematical proof demonstrating that the algorithm will determine whether a magic-labeling exists in solvable cases, and produce a minimal list of criteria for the existence of solutions in the non-solvable cases. This outcome will be considered "highly successful" if I can also prove the algorithm to have "polynomial run-time." In layman's terms, polynomial run-time means that as the size of the graphs we study grows larger, the time necessary to run the algorithm will not increase so fast as to be problematic.

I have already completed much of this portion. I've identified the non-solvable cases for cyclical groups or products of cyclical groups – most groups which arise in practical applications are isomorphic (i.e. have the same structure and properties) with these groups. In such cases, what happens is that one or more equations contains a lead coefficient which isn't a divisor of the identity element. These equations can be treated as a sub-system, and put into a reduced form, to create a minimal list of criteria whose simultaneous satisfaction indicates a solution. I have yet to analyze the run-time of my algorithm as this will depend on the efficiency of my programmer. I can guarantee this first outcome will be "successful," and I am optimistic about it eventually being "highly successful."

The second outcome is the encoding of the algorithm in C++ (or another similar, suitable language). It will be assessed as "successful" if the code will determine whether graphs have magic labelings for cyclical groups and products of cyclical groups. It will be assessed as "highly successful" if we can generalize it to any finite, abelian group ("abelian" means the group operation is commutative – this is a necessary property for labeling problems of this kind).

The third outcome is that the use of the algorithm will produce publishable research results in the area of group-magic labeling. The outcome will be assessed as "successful" if we can publish a paper on the algorithm itself, demonstrating it to be a useful tool for mathematicians to use in graph theory explorations. It will be assessed as "highly successful" if, as intended from the beginning, we manage to completely categorize which graphs have group-magic labeling for one or more non-trivial groups. This is, of course, the ultimate goal of our research.

6. Please describe any planned follow-up activities and how they relate to the goal and purpose of the research/scholarship.

After presenting our algorithm at conferences and making our software openly accessible on the web, we have two possible paths, both of which I would like to pursue.

First, it will probably an immensely difficult task to completely categorize all graphs or all abelian groups, assuming this is even possible. In all likelihood, there will be many years of discoveries and publishable research ahead before the mountaintop, however high it may be, is reached.

Second, there are a variety of other graph theory research areas for which our generalized Gaussian elimination software could be useful. I'm aware of different graph labeling open problems that we could explore, and I expect we will learn about more from other conference-goers.

7. Please describe the relevant experience of the principal investigator related to this research/scholarship project and how the experience will enhance the project.

I am well-published in graph theory, particularly with regard to the study of disjoint path connection in interconnection networks. References **[3]**, **[4]**, and **[5]** are my three most recent publications – in particular, **[3]** and **[4]** detail the "Nova Graph," which I invented and proved to be the optimal construction for simultaneous communications (and hence faster computation speed) in certain parallel processing applications. In the next section, I give a brief explanation of why this project relates to interconnection networks.

I have also written a textbook on linear algebra **[1]**, which was the crux of my doctoral dissertation – this is what I received my doctorate from Carnegie Mellon for. Chapter 3 of my textbook covers the Gaussian elimination procedure in meticulous detail. The algorithm I've created for this research project will involve generalizing this procedure.

Though I'm capable of pursuing this research on my own, I expect several of my colleagues will join in on the exploration. We have a small graph theory research group at UDM, and have even hosted two regional conferences on this subject – I co-organized one of them.

8. Please provide a brief description of the field of discipline and how this research/scholarship will benefit the discipline.

Graph theory is possibly the most versatile of all areas of mathematics when it comes to applications, because graphs are the skeletons of so many mathematical models. In most of my previous research, graphs were used to model interconnection networks, with vertices (points) representing computers and processors, and edges (connecting curves) representing paths of communication. Our past research is useful in designing faster computing, as it describes how to route simultaneous communications within any sized network, from the hardwiring of a microchip to worldwide linking of computers for parallel processing.

Wallis' monograph on magic labeling **[2]** details the importance of magic labeling in interconnection networks – address labeling. Each communication in a network involves three components – a transmitting processor, a receiving one, and a path of communication. The idea is to apply a series of labels to each processor and each path of communication, so that for any information transmission during network operation, if only two of these are known, the third can be deduced via simple arithmetic. Only a magic labeling has this property, enabling a more secure, reliable, and traceable flow of information. Along similar lines, recent research **[6] [7]** has unearthed cryptography applications. This brings into play the possibility of soliciting a future grant from the National Security Agency or Homeland Security.

9. Describe any plans for publication, presentation, or other scholarly outcomes.

First of all, the algorithm and program which runs it will become "Share-Ware" – we will offer it freely to any scientists who might wish to use it in their research pursuits. We will publish a paper on the algorithm in a reputable field journal, most likely ***Congressus Numerantium*** or the ***Journal of Combinatorial Mathematics and Combinatorial Computing*** (JCMCC). A sister article in the ***American Journal of Undergraduate Research*** (AJUR) is very likely, as I am a regional editor of it. If our results are sufficiently significant, I will consider a submission to ***Ars Combinatoria***.

We will also present our results at upcoming Midwestern Graph Theory conferences. These are regional conferences in the Great Lakes area – the UDM Graph Theory Research Group had the privilege of hosting the 45th Midwestern Graph Theory conference in October of 2007. Pending additional future funding, I will also present to the American Mathematical Society (AMS) and the Society of Industrial and Applied Mathematics (SIAM).

Lastly, I will pursue an FGIP grant in the College of Engineering and Science, to fund continued work by undergraduate mathematics and computer science majors along this and related lines of research.

10. Please provide a work plan for each term of the grant period (fall, winter, summer) including specific objectives to be completed in each term and the person(s) responsible.

FALL 2012:

- Theoretical development of the algorithm (**Jeffe Boats**)
- Programming and debugging of the algorithm (**Jeffe Boats + Math-CSSE colleagues**)
- Testing the algorithm using known results from research literature (**Jeffe Boats**)
- Identify two undergraduate research assistants for the Winter term (**Jeffe Boats, with the assistance of some previous faculty instructors of MTH4020 and MTH4600**)

WINTER 2013:

- Write research paper on use of algorithm to establish group-magic results (**Jeffe Boats**)
- Run algorithm to collect and store vast data sets about which graphs have group-magic labelings, and for which groups (**Jeffe Boats + Math-CSSE colleagues**)
- Guide students in exploration of data sets, looking for patterns that can be explored theoretically (**Jeffe Boats + undergraduate research assistants**)
- Presentation of algorithm and results at the 43rd Southeastern International Conference on Combinatorics, Graph Theory, and Computing, and the 54th Midwestern Graph Theory conference (**Jeffe Boats**)

SUMMER 2013:

- Continued investigation of data sets, toward classifications (**Jeffe + Math-CSSE colleagues**)
- Preparation of multiple research papers on results, from both mathematical and computing perspectives (**Jeffe Boats**)
- Determination of next directions based on outcomes (**Jeffe Boats**)
- Drafting of terminal report for this mini-grant (**Jeffe Boats**)